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## Master’s Presentation

Simon Foldvik, Autumn 2024

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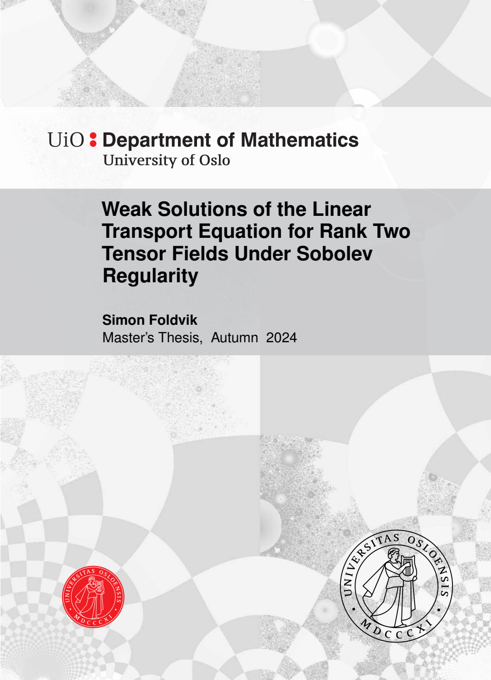

# Topic

The transport equation for rank two tensor fields:

$$\begin{cases} \big(\partial_t + \mathscr{L}_V\big)g = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^n \\ \qquad g(0,\cdot) = g_0 \end{cases}.$$

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# Main Result

Given:

1. Vector field $V$ in $H^1$.
2. Initial rank two tensor field $g_0$ in $L^2$.
3. Final time $T > 0$.

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3. Final time $T > 0$.

If $\operatorname{Sym} DV \in L^\infty$, then there exists in $L^\infty\big(0, T; L^2T^2(\mathbb{R}^n)\big)$ a weak solution $g$ of:

$$\begin{cases} \big(\partial_t + \mathscr{L}_V\big)g = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^n \\ \qquad g(0, \cdot) = g_0 \end{cases}.$$

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# Insight

Control of $\operatorname{Sym} DV$ important for well-posedness?

# Literature Overview

- DiPerna & Lions [2]: Scalar transport equation.
- Heumann *et al.* [4]: Extensions to transport of differential forms.
- Ambrosio [1]: Flows of non-smooth vector fields and applications.
- This thesis [3]: Transport of symmetric tensor fields of rank two.

# Thesis Outline

Chapters:

- Chapter 1 — Introduction
- Chapter 2 — Preliminaries
- Chapter 3 — Scalar Transport Equation
- Chapter 4 — Tensor Transport Equation
- Chapter 5 — Future Work
- Appendix A — Mollifiers

# About

- Author: Simon Foldvik
- Supervisor: Professor Snorre Harald Christiansen
- Scope: 60 ECTS credits
- Period: June–October 2024

# Contents

# Main Result

# The Main Result

Theorem (4.5.2)

*If* $V \in H^1(\mathbb{R}^n)^n$ *and* $\operatorname{Sym} DV \in L^\infty$*, then there exists a weak solution to*

$$\left(\partial_t + \mathscr{L}_V\right) g = 0 \quad \textit{in} \quad (0, T) \times \mathbb{R}^n$$

*in* $L^\infty\left(0, T; L^2T^2(\mathbb{R}^n)\right)$ *starting at any* $g_0 \in L^2T^2(\mathbb{R}^n)$*.*

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Outline as in DiPerna & Lions [2, Proposition II.1], but needs new techniques:

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1. Consider the smoothed problems

$$\begin{cases} \left(\partial_t + \mathscr{L}_{V^{(\epsilon)}}\right) g^{(\epsilon)} = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^n \\ g_0^{(\epsilon)} = g_0 * \phi_\epsilon \end{cases} \qquad (\epsilon > 0)$$

   for a mollifier family $(\phi_\epsilon)_{\epsilon>0}$.

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2. Obtain family $\left(g^{(\epsilon)}\right)_{\epsilon>0}$ of strong solutions bounded in $L^\infty\left(0,T; L^2 T^2(\mathbb{R}^n)\right)$.

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2. Obtain family $\left(g^{(\epsilon)}\right)_{\epsilon>0}$ of strong solutions bounded in $L^\infty\left(0,T; L^2T^2(\mathbb{R}^n)\right)$.

3. Obtain weak solution as a weak$^*$-subsequential limit of $\left(g^{(\epsilon)}\right)_{\epsilon>0}$.

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# Notation

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- Frobenius inner product on rank two tensor fields:

$$\left\langle g, h \right\rangle_{L^2 T^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \left\langle g_x, h_x \right\rangle_{\mathrm{Tr}} \, dx = \sum_{ij} \int_{\mathbb{R}^n} g_{ij} h_{ij} \, dx.$$

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# Smoothed Problems

Smoothed problems with conditions as in Theorem 4.5.2:

$$\begin{cases} \left(\partial_t + \mathscr{L}_{V^{(\epsilon)}}\right) g^{(\epsilon)} = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^n \\ g_0^{(\epsilon)} = g_0 * \phi_\epsilon \end{cases} \qquad (\epsilon > 0).$$

Then:

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Then:

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Then:

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- $g_0^{(\epsilon)}$ is $C^\infty$ and $L^2$ (Lemma 2.3.7),

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Then:

- $V^{(\epsilon)}$ is $C^\infty$ and Lipschitz (Lemma 3.4.4),
- $g_0^{(\epsilon)}$ is $C^\infty$ and $L^2$ (Lemma 2.3.7),
- There exists a unique strong solution $g^{(\epsilon)}$ in $C^\infty(\mathbb{R}^{n+1})$ (Proposition 4.1.1).

# Energy Estimates

## Corollary (4.3.7)

*If $g$ is the unique $C^1$ strong solution to*

$$\big(\partial_t + \mathscr{L}_V\big)g = 0 \quad \mathit{in} \quad \mathbb{R} \times \mathbb{R}^n$$

*starting at $g_0 \in (C^1 \cap L^2)T^2(\mathbb{R}^n)$ for $V$ Lipschitz and $C^2$, then:*

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$$\int_{\mathbb{R}^n} \operatorname{Tr}\big(g_t^T g_t\big)\, dx \le \exp\big(C \|\operatorname{Sym} DV\|_{L^\infty T^2} t\big) \int_{\mathbb{R}^n} \operatorname{Tr}\big(g_0^T g_0\big)\, dx \quad (t \ge 0).$$

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*In particular:* $g_t$ *remains* $L^2$ *for all later times* $t \ge 0$*.*

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1. Functional density estimation (§§ 4.2, 4.3):

$$\left(\partial_t + V \cdot \nabla\right) \operatorname{Tr}\left(g_t^T g_t\right) = -2 \operatorname{Tr}\left(\left(g_t g_t^T + g_t^T g_t\right) DV\right).$$

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2. Exact evolution of $L^2$-energies (Proposition 4.3.6):

$$\partial_t \int_{\mathbb{R}^n} \operatorname{Tr}\left(g_t^T g_t\right) dx = \int_{\mathbb{R}^n} \operatorname{div}(V) \operatorname{Tr}\left(g_t^T g_t\right) - 2 \int_{\mathbb{R}^n} \operatorname{Tr}\Big(\left(g_t g_t^T + g_t^T g_t\right) \operatorname{Sym} DV\Big)\, dx.$$

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# Energy Estimates

Strong solutions thus satisfy uniform $L^2$-bounds:

$$\sup_{0 \le t \le T} \|g_t\|_{L^2T^2} \le C_T \|g_0\|_{L^2T^2}$$

with $C_T := \exp\big(C\|\operatorname{Sym} DV\|_{L^\infty T^2} T\big)$.

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with $C_T := \exp\big(C\|\mathrm{Sym}\, DV\|_{L^\infty T^2} T\big)$.

Hence $g_\bullet \in L^\infty\big(0, T; L^2T^2(\mathbb{R}^n)\big)$.

# Boundedness of Smoothed Solutions

With $g^{(\epsilon)}$ solving

$$\begin{cases} \left(\partial_t + \mathscr{L}_{V^{(\epsilon)}}\right) g^{(\epsilon)} = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^n \\ g_0^{(\epsilon)} = g_0 * \phi_\epsilon \end{cases} \qquad (\epsilon > 0)$$

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strongly with conditions as in Theorem 4.5.2:

$$\left\| g_\bullet^{(\epsilon)} \right\|_{L^\infty(0,T;L^2T^2)} \leq \exp\left( C \| \operatorname{Sym} DV^{(\epsilon)} \|_{L^\infty T^2} T \right) \| g_0 * \phi_\epsilon \|_{L^2T^2} \quad (\epsilon > 0).$$

# Boundedness of Smoothed Solutions

$\left(g^{(\epsilon)}\right)_{\epsilon>0}$ is bounded in $L^\infty\left(0, T; L^2T^2(\mathbb{R}^n)\right)$:

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# Boundedness of Smoothed Solutions

$\left(g^{(\epsilon)}\right)_{\epsilon>0}$ is bounded in $L^\infty\left(0, T; L^2T^2(\mathbb{R}^n)\right)$:

$$\|g_0 * \phi_\epsilon\|_{L^2T^2} \leq C\|g_0\|_{L^2T^2},$$

$$\|\operatorname{Sym} DV^{(\epsilon)}\|_{L^\infty T^2} = \|(\operatorname{Sym} DV) * \phi_\epsilon\|_{L^\infty T^2} \leq \|\operatorname{Sym} DV\|_{L^\infty T^2}.$$

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# Compactness Criterion

## Lemma (4.5.4)

*Bounded sequences in* $L^\infty\left(0, T; L^2T^2(\mathbb{R}^n)\right)$ *are sequentially precompact for the weak*$^*$*-topology.*

Subtle…

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Subtle...

## Lemma (3.4.6)

*Bounded sequences in the dual* $F^*$ *of a separable Banach space* $F$ *are sequentially precompact for the weak*$^*$*-topology.*

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# Bochner Duality

Isometric isomorphism (Frobenius inner product):

$$L^\infty\big(0, T; L^2T^2(\mathbb{R}^n)\big) \simeq L^1\big(0, T; L^2T^2(\mathbb{R}^n)\big)^*.$$

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- $L^2T^2(\mathbb{R}^n) \simeq \bigoplus_1^{n^2} L^2(\mathbb{R}^n)$ is separable.
- $L^1\big(0, T; L^2T^2(\mathbb{R}^n)\big)$ is separable [5, Proposition 1.2.29].

# Subsequence Extraction

There is a subsequence $\left(g^{(k)}\right)_{k\in\mathbb{N}}$ with $g^{(k)} \to g$ in $L^\infty\left(0, T; L^2T^2(\mathbb{R}^n)\right)$ weakly$^*$:

$$\int_0^T \left\langle g_t^{(k)}, \Psi_t \right\rangle_{L^2T^2} dt \to \int_0^T \left\langle g_t, \Psi_t \right\rangle_{L^2T^2} dt$$

for all $\Psi \in L^1\left(0, T; L^2T^2(\mathbb{R}^n)\right)$.

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for all $\Psi \in L^1\left(0, T; L^2T^2(\mathbb{R}^n)\right)$.

**Claim:** This $g$ is a weak solution.

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Generalisation of definitions in DiPerna & Lions [2, p. 514]:

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- $\mathscr{D}_T(E)$: Smooth test fields $\Phi\colon [0,T] \times \mathbb{R}^n \to E$ with

$$\operatorname{supp}\Phi \subset\subset [0,T) \times \mathbb{R}^n.$$

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# Variational Formulation — Test Fields

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$$\operatorname{supp}\Phi \subset\subset [0,T)\times\mathbb{R}^n.$$

- § 3.2: Regularity properties of test fields $\Phi \in \mathscr{D}_T(E)$, e.g.,

$$\Phi_\bullet\colon t \mapsto \Phi_t \text{ Lipschitz } [0,T] \to L^p(\mathbb{R}^n, E) \quad (1 \le p \le \infty).$$

# Variational Formulation — Tensor

Strong solutions satisfy a variational equation (Proposition 4.4.1):

$$
\begin{aligned}
-\int_0^T \left\langle g_t, (\partial_0\Phi)_t \right\rangle_{L^2T^2} dt - \int_0^T \left\langle g_t, \mathscr{L}_V \Phi_t \right\rangle_{L^2T^2} dt \\
- \int_0^T \left\langle g_t, \operatorname{div}(V)\Phi_t \right\rangle_{L^2T^2} dt \\
+ 2\int_0^T \left\langle g_t, \operatorname{Sym}(DV)\Phi_t \right\rangle_{L^2T^2} dt \\
+ 2\int_0^T \left\langle g_t, \Phi_t \operatorname{Sym}(DV) \right\rangle_{L^2T^2} dt = \left\langle g_0, \Phi_0 \right\rangle_{L^2T^2}
\end{aligned}
$$

for all $\Phi \in \mathscr{D}_T(\operatorname{Bil}\mathbb{R}^n)$.

# Variational Formulation — Scalar

Compare with variational formulation for the scalar transport equation (§ 3.3):

$$-\int_0^T \left\langle u_t, (\partial_0 \phi)_t \right\rangle_{L^2} dt - \int_0^T \left\langle u_t, \mathrm{div}(V)\phi_t \right\rangle_{L^2} dt \\ - \int_0^T \left\langle u_t, (V \cdot \nabla)\phi_t \right\rangle_{L^2} dt = \left\langle u_0, \phi_0 \right\rangle_{L^2}$$

for all $\phi \in \mathscr{D}_T(\mathbb{R})$.

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# Variational Formulation — Tensor

Introduce (§ 4.4):

$$
\begin{aligned}
B_1(g,\Phi) &:= \int_0^T \left\langle g_t, (\partial_0 \Phi)_t \right\rangle_{L^2T^2} dt, \\
B_2(g,\Phi) &:= \int_0^T \left\langle g_t, \mathscr{L}_V \Phi_t \right\rangle_{L^2T^2} dt, \\
B_3(g,\Phi) &:= \int_0^T \left\langle g_t, \operatorname{div}(V)\Phi_t \right\rangle_{L^2T^2} dt, \\
B_4(g,\Phi) &:= -2\int_0^T \left\langle g_t, \operatorname{Sym}(DV)\Phi_t \right\rangle_{L^2T^2} dt, \\
B_5(g,\Phi) &:= -2\int_0^T \left\langle g_t, \Phi_t \operatorname{Sym}(DV) \right\rangle_{L^2T^2} dt.
\end{aligned}
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# Weak Solutions (Definition 4.5.1)

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Is $g \in L^\infty(0,T; L^2T^2(\mathbb{R}^n))$ solving

$$-\sum_{j=1}^{5} B_j(g,\Phi) = \langle G, \Phi_0 \rangle_{L^2T^2} \quad \text{for all} \quad \Phi \in \mathscr{D}_T(\mathrm{Bil}\,\mathbb{R}^n).$$

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# Existence of Weak Solutions

Sequence $\left(g^{(k)}\right)_{k\in\mathbb{N}}$ of smoothed problems solve:

$$-\sum_{j=1}^{5} B_j^{(k)}\left(g^{(k)}, \Phi\right) = \left\langle G * \phi_k, \Phi_0\right\rangle_{L^2T^2} \quad (\Phi \in \mathscr{D}_T(\mathrm{Bil}\,\mathbb{R}^n)).$$

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# Existence of Weak Solutions

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Send $k \longrightarrow \infty$ to obtain (recalling $g^{(k)} \longrightarrow g$ weakly$^*$):

$$-\sum_{j=1}^{5} B_j(g, \Phi) = \left\langle G, \Phi_0\right\rangle_{L^2T^2} \quad (\Phi \in \mathscr{D}_T(\mathrm{Bil}\,\mathbb{R}^n)).$$

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# Conclusion

This concludes the proof of Theorem 4.5.2.

# Contents

# Contributions

# Chapter 2 — Preliminaries

Main contributions in Chapter 2:

- Distributions and convolution of maps with values in vector spaces.
- Weak advection operator on vector fields in $L^p$ for $1 \leq p < \infty$.
- Lie decomposition formula.
- Lie adjoint formula.
- Lie derivative on Sobolev vector fields.

# Chapter 3 — Scalar Transport Equation

Main contributions in Chapter 3:

- Explicit determination of $L^p$-bounds on strong solutions.
- Generalised energy estimates (‘$\phi$-energies’).
- Energy equivalence principle.
  - Application: Exponential bound on Jacobian of flow.
  - Application: Exponential bound on growth of volumes under the flow.
- Temporal $L^p$-regularity of strong solutions.
- Temporal regularity considerations of test fields in $\mathscr{D}_T(E)$.
- Functional analytic principles for weak$^*$-convergence.
- Smoothed Sobolev vector fields are also Lipschitz.

# Chapter 4 — Tensor Transport Equation

Main contributions in Chapter 4:

- Energy transport relation for non-linear functionals.
  - Application: Functional density estimation technique.
  - Application: Energy estimates (transport of $L^2$-density).
  - Application: Control of $\operatorname{Tr} g_t$, $\det g_t$, etc.
- Energy estimates for strong solutions in $C^1 \cap L^2$.
- Exact relation for the evolution of $L^2$-energies of strong solutions.
- Variational formulation.
- Suggests a notion of weak solution when $V$ is $H^1$.
- Existence result for weak solutions in $L^\infty(0, T; L^2T^2(\mathbb{R}^n))$.

# Highlight

We will take a closer look at these:

1. Lie derivative on $H^1$.
2. Energy equivalence principle.
3. Energy transport relation.

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# The Lie Derivative

## The Lie Derivative on Sobolev Vector Fields

# Lie Adjoint

Frobenius adjoint of the Lie derivative (Proposition 2.7.1):

$$\left\langle \mathscr{L}_V g, \Phi \right\rangle_{L^2 T^2} = \left\langle g, \mathscr{L}_V^* \Phi \right\rangle_{L^2 T^2},$$

where

$$\mathscr{L}_V^* \Phi = -\mathscr{L}_V \Phi - \operatorname{div}(V)\Phi + 2\operatorname{Sym}(DV)\Phi + 2\Phi\operatorname{Sym}(DV).$$

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where

$$\mathscr{L}_V^* \Phi = -\mathscr{L}_V \Phi - \operatorname{div}(V)\Phi + 2\operatorname{Sym}(DV)\Phi + 2\Phi\operatorname{Sym}(DV).$$

Note: $\mathscr{L}_V^* \Phi$ is symmetric if $\Phi$ is symmetric.

# Lie Adjoint

Frobenius adjoint of the Lie derivative (Proposition 2.7.1):

$$\left\langle \mathscr{L}_V g, \Phi \right\rangle_{L^2T^2} = \left\langle g, \mathscr{L}_V^* \Phi \right\rangle_{L^2T^2},$$

where

$$\mathscr{L}_V^* \Phi = -\mathscr{L}_V \Phi - \operatorname{div}(V)\Phi + 2\operatorname{Sym}(DV)\Phi + 2\Phi\operatorname{Sym}(DV).$$

Note: $\mathscr{L}_V^* \Phi$ is symmetric if $\Phi$ is symmetric.

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Note: $\mathscr{L}_V + \mathscr{L}_V^*$ is of order zero.

# Lie Derivative on $H^1$

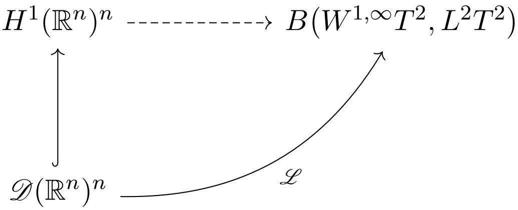


Figure: Extending the Lie derivative from smooth fields with compact support to $H^1$.

# Lie Derivative on $H^1$

Extension procedure (§ 2.8):

1. For $V \in \mathscr{D}(\mathbb{R}^n)^n$ and $g \in W^{1,\infty}T^2$: Exists unique $\mathscr{L}_V g \in L^2T^2$ such that:

$$\left\langle \mathscr{L}_V g, \Phi \right\rangle_{L^2T^2} = \left\langle g, \mathscr{L}_V^* \Phi \right\rangle_{L^2T^2}$$

   for all $\Phi \in \mathscr{D}T^2(\mathbb{R}^n)$.

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3. The map $V \mapsto \mathscr{L}_V$ is $H^1$-bounded

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4. There is a unique extension to $H^1(\mathbb{R}^n)^n$.

# Energy Equivalence Principle

ENERGY EQUIVALENCE PRINCIPLE

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Theorem (3.1.13)

*Energy estimates in $L^p$ are equivalent with $L^\infty$-bounds on the Jacobian of the flow:*

$$\left\{\|\mathrm{Jac}\,\Phi_t\|_{L^\infty(\mathbb{R}^n)} \le c(t)\right\} \simeq \left\{\int_{\mathbb{R}^n} |u_t|^p \, dx \le c(t) \int_{\mathbb{R}^n} |u_0|^p \, dx\right\} \quad (1 \le p < \infty)$$

*for strong solutions of the scalar transport equation:*

$$\big(\partial_t + V \cdot \nabla\big) u = 0 \quad in \quad \mathbb{R} \times \mathbb{R}^n.$$

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# Energy Transport Relation (§ 4.2)

## ENERGY TRANSPORT RELATION

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If $g$ is the $C^1$ strong solution to

$$\big(\partial_t + \mathscr{L}_V\big)g = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^n$$

and $\Psi\colon \ \mathrm{Bil}(\mathbb{R}^n) \to \mathbb{R}$ is a $C^1$ functional, then (Proposition 4.2.1):

$$\big(\partial_t + V \cdot \nabla\big)\Psi(g_t) = -D\Psi(g_t)\big((DV)^T g_t + g_t(DV)\big).$$

# Energy Transport Relation (§ 4.2)

Integrating the energy transport relation by parts:

$$\partial_t \int_{\mathbb{R}^n} \Psi(g_t)\, dx - \int_{\mathbb{R}^n} \operatorname{div}(V)\Psi(g_t)\, dx = -\int_{\mathbb{R}^n} D\Psi(g_t)\big((DV)^T g_t + g_t(DV)\big)\, dx.$$

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# Energy Transport Relation (§ 4.2)

*Functional density estimation*:

$$\left|D\Psi(g_t)\left((DV)^T g_t + g_t(DV)\right)\right| \leq c(t)\Psi(g_t)$$

for some control $c\colon \mathbb{R} \to \mathbb{R}_+$.

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Leads to:

$$\partial_t \int_{\mathbb{R}^n} \Psi(g_t)\, dx \leq \big(C_0 + c(t)\big) \int_{\mathbb{R}^n} \Psi(g_t)\, dx.$$

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# Energy Transport Relation (§ 4.2)

## Example (4.2.3)

With $\Psi(B) := \operatorname{Tr}\left(B^T B\right)$,

$$\left(\partial_t + V \cdot \nabla\right) \operatorname{Tr}\left(g_t^T g_t\right) = -2 \operatorname{Tr}\Big(\left(g_t g_t^T + g_t^T g_t\right) DV\Big),$$

using $D\Psi(B)H = 2 \operatorname{Tr}\left(B^T H\right)$.

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Leads to energy estimates in $L^2$.

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# Energy Transport Relation (§ 4.2)

## Example (4.2.5)

With $\Psi := \operatorname{Tr}$,

$$\big(\partial_t + V \cdot \nabla\big) \operatorname{Tr} g_t = -2 \operatorname{Tr}\big(\operatorname{Sym}(DV) g_t\big),$$

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## Example (4.2.5)

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using $D\Psi(B) = \mathrm{Tr}$.

If $\operatorname{Sym} DV = 0$:

$$\big(\partial_t + V \cdot \nabla\big) \operatorname{Tr} g_t = 0.$$

# Energy Transport Relation (§ 4.2)

Clearly not exhausted the potential of this technique.

Perhaps useful for uniqueness of weak solutions?

# Transport of the determinant

Energy transport relation suggests:

$$\left(\partial_t + V \cdot \nabla\right) \det g_t = -2 \operatorname{div}(V) \det g_t.$$

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With $\Psi := \det$:

$$D\Psi(B)H = \det(B) \operatorname{Tr}\left(B^{-1}H\right) \quad (B \in \mathrm{GL}_n(\mathbb{R}), H \in \mathcal{M}_n(\mathbb{R})).$$

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Energy transport relation:

$$\big(\partial_t + V \cdot \nabla\big) \det(g_t) = -\det(g_t) \operatorname{Tr}\Big(g_t^{-1}(DV)^T g_t + g_t^{-1} g_t (DV)\Big). \qquad \blacksquare$$

# Transport of determinant

Integrate $\left(\partial_t + V \cdot \nabla\right) \det g_t = -2\,\mathrm{div}(V) \det g_t$:

$$\partial_t \int_{\mathbb{R}^n} \det g_t \, dx = -\int_{\mathbb{R}^n} \mathrm{div}(V) \det g_t \, dx.$$

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If $\operatorname{div} V = 0$:

- $\big(\partial_t + V \cdot \nabla\big) \det g_t = 0$,
- $\det g_t(x) = \det g_0\big(\Phi_{-t}(x)\big)$,
- $\int \det g_t \, dx = \int \det g_0 \, dx$.

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# Further Results

# Some results did not make it into the thesis.

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1. New example that pointwise limits of measurables may fail to be measurable.

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1. New example that pointwise limits of measurables may fail to be measurable.

2. New proof that pointwise limits of measurables are measurable when valued in spaces second countable and regular (Tychonoff cube):

$$(M, \mathcal{A}) \xrightarrow{u} X \hookrightarrow \prod_{n\in\mathbb{N}} [0,1]_n \xrightarrow{\pi_k} [0,1]_k$$

and

$$\mathscr{B}\Big(\prod_{n\in\mathbb{N}} [0,1]_n\Big) \subseteq \bigotimes_{n\in\mathbb{N}} \mathscr{B}([0,1]_n).$$

# Future Work

# Future Work

- Properties of weak solutions (symmetry, positivity, ...)?
- Sharp conditions for existence?
- Uniqueness (renormalised solutions, energy transport relation, ...)?
- Compare with flow techniques of weakly differentiable vector fields in Ambrosio [1] and references therein.

# References I

Unqualified references (‘Theorem 4.5.2’) are to [3].

[1] Luigi Ambrosio. ‘Well posedness of ODE’s and continuity equations with nonsmooth vector fields, and applications’. In: *Revista Matemática Complutense* 30.3 (2017), pp. 427–450.

[2] R. J. DiPerna and P. L. Lions. ‘Ordinary differential equations, transport theory and Sobolev spaces’. In: *Inventiones mathematicae* 98.3 (Oct. 1989), pp. 511–547.

[3] Simon Foldvik. ‘Weak Solutions of the Linear Transport Equation for Rank Two Tensor Fields Under Sobolev Regularity’. Master’s thesis. University of Oslo, 2024.

# References II

[4] H. Heumann, Ralf Hiptmair and Cecilia Pagliantini. ‘Stabilized Galerkin for transient advection of differential forms’. In: *Discrete and Continuous Dynamical Systems Series S* 9.1 (Feb. 2016), pp. 185–214.

[5] Tuomas Hytönen et al. *Analysis in Banach Spaces. Volume I: Martingales and Littlewood–Paley Theory*. First Edition. A Series of Modern Surveys in Mathematics 63. Cham, Switzerland: Springer, 2016.

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**Master's Presentation**
Simon Foldvik, Autumn 2024